Boundary Conditions that Imitate Cauchy Problem for Finite-Difference Approximations of Basic Mathematical Physics Equations

V.A.Gordin, A.A.Shemendyuk

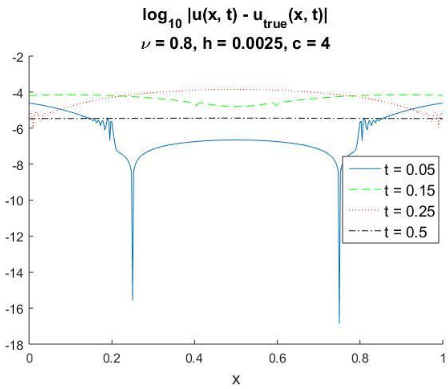
National Research University "Higher School of Economics",
Hydrometeorological Center of Russia

Most common physical processes can be described by evolutionary partial differential equations or systems. To calculate a solution of such problems, finite-difference schemes may be used. However, due to a finite computer memory and a computational time, solving the problem in an infinite area is impossible. At the same time, a mixed boundary problem may be solved. Usually boundary conditions describe some physical processes that occur on the area's boundary. For example, the Dirichlet boundary condition simulates solution's fixation on the boundary; the Neuman boundary condition simulates the fixation of the first normal (with respect to the boundary) derivative. Sometimes, there are no such special processes happening on the boundary. However, usual boundary conditions will reflect some waves, which in nature go away from the computational domain. The good solution of the mixed value problem should be the same as the solution for the Cauchy's problem.

The boundary conditions, which provide the solution that coincide with the Cauchy's problem solution (ICP boundary conditions) for important linear differential equations and systems with constant coefficients, exist and are nonlocal. They include integral operators like convolution. As about finite difference case we should use in the convolutions series instead of integrals. Common algorithm of building such conditions for half-space was provided in [1]. Since ICP boundary conditions are nonlocal, all solution's values in the previous time moments are required to calculate next boundary values. The Hermite – Pade approximations of the symbols of the finite-difference operators are used [2] to reduce a number of required arithmetical operations and volume of memory.

We consider few basic equations of mathematical physics: wave, diffusion and Schrödinger equations. For these equations some finite-difference approximations were considered (leap-frog, implicit Euler scheme, Crank – Nicolson, compact schemes). In every case, different ICP boundary conditions are obtained. For diffusion and Schrödinger equations, it is shown that localization of the ICP boundary conditions leads to the growth of the solution's error (difference between the exact solution and the obtained one).

As an example, let us consider one–dimensional wave equation $\partial_t^2 u = c^2 \partial_x^2 u$, where c – velocity of the wave, and approximate if with a leap-frog scheme with a space step h and a time step τ. Figure 1 shows logarithm of absolute difference between an exact solution u_{true} and the obtained one u at different slices of time ($\nu = c\tau/h$ - Courant's parameter).



[1] V. Gordin, "About Mixed Boundary Problem that Imitates Cauchy's Problem," *Uspekhi Matematicheskikh Nauk*, vol. 33, no. 5, pp. 181-182, 1978.

[2] V. Gordin, Mathematics, computer, weather forecast and other scenaria of mathematical physics, Moscow: FIZMATLIT, 2010, 2013.

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