# Singular solutions for vibration control problems

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We consider the control system

$$\frac{\partial^2}{\partial t^2} p(x,t) + A_x p(x,t) = g(x,t), \quad x \in G, \quad t \in \mathbb{R}_+ \tag{1}$$

subject to the initial

$$p(x,0) = v_0(x), \quad p_t(x,0) = v_1(x) \tag{2}$$

and the boundary conditions

$$B_x p|_{\partial G} = 0 \tag{3}$$

Here, $p(x,t)$ and $g(x,t)$ are $m$-dimensional vector-valued functions defined on $G \times [0,\infty)$, $G$ is a compact subset of $\mathbb{R}^n$ with smooth boundary $\partial G$, $A_x$ is a linear elliptic operator of order 2 in $G$, $B_x$ is a linear differential operator of order 1 in a neighborhood of $\partial G$. Suppose that $p(x,t) \in L_2(G \times [0,\infty), \mathbb{R}^m)$, $g$ is a measurable function in $t$, $g(\cdot,t) \in L_2(G,\mathbb{R}^m)$, $p(\cdot,t)$, $\frac{\partial^2}{\partial t^2} p(\cdot,t)$, $v_0(x), v_1(x) \in H^2(G,\mathbb{R}^m)$.

The external force $g(x,t)$ is considered as a control function. We assume that $g(x,t)$ is a bounded:

$$\|g(\cdot,t)\|^2_{L_2(G,\mathbb{R}^m)} \leq 1 \tag{4}$$

We study the following control problem: to find the admissible control such that the coresponding solution of (1)-(3) minimize the functional

$$\int_0^\infty \|p(\cdot,t)\|^2_{L_2(G,\mathbb{R}^m)} dt \to \inf \tag{5}$$

Assume that $A_x$ with domain $D_A = \{y \in H^2(G,\mathbb{R}^m), \quad Bp|_{\partial G} = 0\}$ is a self-adjoint operator, has the discrete spectrum $\{\lambda_j\}_{j=1}^\infty$ and the orthonormal complete system of eigenfunctions $\{h_j(x)\}_{j=1}^\infty$ that are smooth in $G$ and satisfying (3). We seek a solution of (1)-(5) in the form

$$p(x,t) = \sum_{j=1}^\infty q_j(t) h_j(x) \tag{6}$$

where $q_j(t) = (p, h_j)_{L_2(G,\mathbb{R}^m)}, \quad j = 1, 2, \ldots$, are Fourier coefficients. Expand the functions $g, v_0, v_1$ in the basis $\{h_j(x)\}_{j=1}^\infty$

$$g(t,x) = \sum_{j=1}^\infty u_j(t) h_j(x), \quad v_0(x) = \sum_{j=1}^\infty s_j h_j(x), \quad v_1(x) = \sum_{j=1}^\infty r_j h_j(x)$$

We get for the Fourier coefficients $q_j$ the following system of countably many ordinary differential equations: $\ddot{q}_j(t) + \lambda_j q_j(t) = u_j(t)$, $j = 1, 2, \ldots$, with the boundary conditions: $q_j(0) = s_j, \quad \dot{q}(0) = r_j$. Condition (4) gives us $\sum_{j=1}^\infty u_j^2(t) \leq 1$. Substituting (6) into (5) we get

$$\int_0^\infty \|p(\cdot,t)\|^2_{L_2(G,\mathbb{R}^m)} dt = \int_0^\infty \sum_{j=1}^\infty q_j^2(t) dt$$

Thus we have an optimal control problem for the Fourier coefficients in the space $l^2$:

$$\int_0^\infty \sum_{j=1}^{\infty} q_j^2(t)dt \to \min \tag{7}$$

$$\ddot{q}_j(t) + \lambda_j q_j\,(t) = u_j(t) \tag{8}$$

$$q_j\,(0) = s_j, \quad \dot{q}\,(0) = r_j, \quad j = 1, 2, \ldots \tag{9}$$

$$\sum_{j=1}^{\infty} u_j^2\,(t) \le 1 \tag{10}$$

Problem (1)-(5) with the external force in the form $g\,(x,t) = u\,(t)\,f\,(x)$ was considered in [1]. In this case the corresponding control problem for the Fourier coefficients has the same form but (8) and (10) are replaced by

$$\ddot{q}_j(t) + \lambda_j q_j\,(t) = C_j u(t) \tag{11}$$

$$-1 \le u\,(t) \le 1 \tag{12}$$

where $C_j \in \mathbb{R}$. In [2] for the problem (7), (9), (11)-(12) it was constructed the optimal synthesis containing singular extremals and extremals with accumulation of switchings.

In this paper for some initial conditions (2) we prove that the behavior exhibited by the solutions of (7)-(10) is similar to that of the problem with scalar control.

Note that the system (1)-(3) governs vibrations of beams, strings and other mechanical models. We consider, as an example, the problem of controlling the vibrations of the Timoshenko beam.

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# References

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[2] V. Borisov, M. Zelikin, L. Manita, Optimal Synthesis in an Infinite-Dimensional Space, Proceedings of the Steklov Institute of Mathematics 271 (2010) pp. 34-52.