Properties of the Tent map for decimal fractions with fixed precision

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Abstract. A one-dimensional discrete Tent map is a well-known example of a map whose fixed points are all unstable on a segment [0,1]. This map leads to the positivity of the Lyapunov exponent for the corresponding recurrent sequence.
Therefore in a situation of general position, this sequence must demonstrate the properties of deterministic chaos. However if the first term of the recurrence sequence is taken as a decimal fraction with a fixed number "k" of significant digits after the decimal point and all calculations are carried out accurately, then the situation turns out to be completely different. In this case first the Tent map does not lead to an increase in significant digits in the terms of the sequence, and secondly demonstrates the existence of a finite number of eventually periodic orbits, which are attractors for all other decimal numbers with the number of significant digits not exceeding "k".

In this paper we consider the simplest, not containing parameters, Tent map

$$T(x) = 2 \cdot \min\{x, 1-x\} = 1 - 2 \cdot \left|x - 1/2\right|, \qquad (1)$$

and the recurrent sequence

$$x_{n+1} = T(x_n). \qquad (2)$$

Various modifications of this map containing additional parameters can be found both in papers dealing with environmental processes [1,2], and in cryptography [3,4] and in modeling open systems [5,6].

In this paper, the following is shown. Definitions. The set I_k (k = 1,2, ...) is a collection of numbers of the form $0.\alpha_1\alpha_2\alpha_3\ldots,\alpha_k = \sum_{i=1}^{k} \alpha_i \cdot 10^{-i} \in I_k$, where α_i are integers from 0 to 9, and $\alpha_k \neq 0$. The number of elements of this set is $NI_k = 9 \cdot 10^{k-1}$. $I_{k1} \cap I_{k2} = \varnothing \;\; if \;\; k1 \neq k2$. If $J_k = 0 \cup \bigcup_{q=1}^{k} I_q$, then the number of elements of the set J_k is a value $NJ_k = 10^k$.

We call a cycle $S_k(m_k)$ a set of m_k elements $\in I_k$ for which the following property holds: for any $x \in S_k(m_k)$ an image $T(x) \in S_k(m_k)$. The length of the cycle (a number m_k) is determined by the number k by the formula $m_k = 2 \cdot 5^{k-1}$. Statement. For each set I_k there is a unique cycle $S_k(m_k)$.
The union of all cycles contained in a set J_k is the set of numbers $G_k(M_k)$ occurring in all cycles $G_k(M_k) = 0 \cup \bigcup_{q=1}^{k} S_q(m_q)$. The number of elements in the collection of cycles is $M_k = 1 + \sum_{q=1}^{k} m_q = \frac{1}{2}(5^k + 1)$ (odd number). The ratio of the number of these elements to the total number of elements is $\mathrm{M_k/NJ_k} = (2^{-k} + 10^{-k}) \cdot 2^{-1} < 2^{-k}$.
Definition. The sequence of numbers defined by the recurrence relation (2) is called an *orbit* of the map (1), induced by the quantity x_0. We call the orbit as eventually periodic orbit, if the orbit has the structure $x_0, x_1, x_2, \ldots, x_m, x_{m+1}, x_{m+2}, \ldots, x_{m+n}$, and $x_{m+n} = x_m$.

Statement. The set $G_k(M_k)$ consists of eventually periodic orbits and is an attractor for any $x_0 \in J_k$. All the statements are true only when using exact calculations (symbolic calculations) without round-off, which is possible in the package Mathematica.

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